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# DISTRIBUTION OF GAPS AND BLOCKS IN A TRAFFIC STREAM

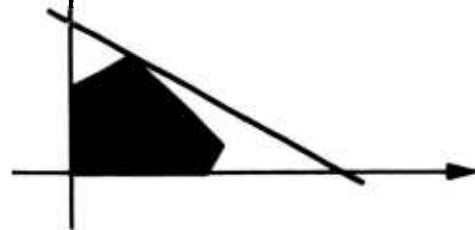
by  
Robert M. Oliver

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UNIVERSITY OF CALIFORNIA - BERKELEY



**DISTRIBUTION OF GAPS AND BLOCKS IN A TRAFFIC STREAM**

by

**Robert M. Oliver  
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**25 July 1961**

**Research Report 10**

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## ABSTRACT

This paper studies some of the theoretical questions of large openings or gaps in a single stream of traffic. A gap in the traffic stream is defined as a headway between vehicles greater than or equal to some minimum size -- say  $x$ . Several authors have studied the probability distribution of the wait which a randomly located observer must endure before he finds a gap. This paper, while briefly reviewing the solutions of this well known problem, is primarily concerned with expressions for: (i) the distribution of gap sizes; (ii) the distribution of spacings between vehicles and gaps; (iii) the mean and variance of inter-vehicle and inter-gap spacings; (iv) the stationary flow rates of gaps; and (v) the distribution of blocked and unblocked periods. It is assumed that the origin of measurements may be located (i) with the passing of a vehicle, (ii) at the beginning of a gap, or (iii) at random. It is also assumed that the distribution of inter-vehicle spacings are independently, but identically, distributed random variables.

## 0. Introduction

There are several well-known variations of a merging problem which arise when traffic in a minor stream joins or crosses a major flow stream, the criterion for merging being that a minimum size gap appear between vehicles in the major stream. As the problem is usually formulated one or all vehicles in the minor stream merge with the major stream whenever the headway to the next vehicle is greater than or equal to the minimum size gap. \*

In automobile traffic this problem arises in low-speed merges when cars come to an intersection, stop and wait for the desired gaps in a cross stream of traffic. While this situation may be one of the more obvious places where a study of gaps in traffic streams is needed there are many other examples: In high-speed automobile merges the question is often one of finding empty road-space into which cars can fit. At airports departing aircraft wait for the use of a runway which also services the higher priority landing aircraft; the former can only occupy the runway when gaps of certain minimum size appear between successive landing aircraft. Flight rules often specify for safety and/or reasons of detectability that, in crossing air-lanes, aircraft at the same altitude must at all times be separated by some minimum headway from one another.

While I have introduced the gaps in the major stream in the context of the wait of a vehicle in the minor stream which looks for the first gap or opening in the major stream, it is at this point that I would like to make clear the distinction between the major and minor stream and clearly identify the fact that the gap production process is one which can go on in the major stream in the absence of a minor stream of crossing or merging traffic.

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\* In earlier papers the word "gap" has often referred to any empty interval; in this paper "gap" specifically refers to intervals between vehicles greater than or equal to  $x$ , the minimum size gap.

The word "minor" certainly seems appropriate when one considers mathematical merging models which allow at most one merging vehicle per gap into the major stream<sup>(10)</sup>. In this case it is easy to show that the stationary flow rate of gaps is always less than the stationary flow rate of vehicles in the major stream; hence the phrase ~~minor~~ stream refers to one whose steady state flow rate is less than that of the major stream. However, in actual merging situations it is often the case that the "minor" stream contributes the larger fraction of downstream flow. Generally speaking, as vehicle flow rates in the major stream decrease, gap sizes increase; many vehicles in the minor stream may be absorbed by a single large gap in the major stream. If, on the average, the total rate of absorption is greater than the "major" stream flow rate it would seem natural to reverse the names of minor and major stream. It is this apparent contradiction which has led me to consider the role of these two streams and to think preferably in terms of a primary and a secondary flow process.

This paper will be primarily concerned with the statistical characteristics of large gaps or openings in a single traffic stream. The flow of vehicles in this major stream, though unpredictable, does not depend on the presence or condition of vehicles in an intersecting or merging stream. Large openings appear in the major stream only because the statistics of inter-vehicle spacings in that stream tell us that one can expect to find vehicles separated by gaps a certain fraction of the time. The casual reader of earlier papers might come to the conclusion that the gap-producing process is a function of the wait endured by a vehicle in the minor stream; rather, it is the other way around, the gap process in the major stream and a specific merging, stopping or crossing mechanism helps to generate a secondary or minor stream process.

The approach used in this paper differs from earlier ones in that the statistical characteristics of the gap-producing process in the major stream are examined and then used to analyze a secondary process (of which there may be many). It is

important to point out, nonetheless, that the observation of the queue length of vehicles or the wait of a single vehicle in a minor stream may be of immense predictive value in determining the appearance of gaps in the major stream.

Assume that inter-vehicle spacings are independently and identically distributed. A certain fraction of the vehicles will be separated by intervals greater than or equal to  $x$  while the remainder will be separated by intervals less than  $x$ . In Section (1) expressions are obtained for the probability distribution of counts of gaps when the distribution of headways between vehicles or gaps is not included in the discussion.

In Section (2) the probability law which describes the length of the interval from the beginning of a gap to the beginning of a successive gap is found as a function of the inter-vehicle distribution and the minimum gap size. The mean and variance of this inter-gap distribution are then related to the mean and variance of the distribution of inter-vehicle headways.

In Section (3) expressions are obtained for the distribution of the size of blocked\* and unblocked periods and the wait for the first unblocked period.

In Section (4) the discrete gap counting distributions and the average and variance of the gap and block flow rate are derived.

In Section (5) analytic and numerical results are given for the case where inter-vehicle headways are exponentially distributed, i. e., vehicle counts are Poisson.

In Section (6) moments of the inter-event distributions are obtained in terms of the vehicle flow rate when vehicles appear in bunches but each vehicle is separated by a minimum headway.

Section (7) compares some of the results obtained in this paper with results of earlier authors.

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\* A blocked period is one in which the headway for any point to the next vehicle to appear is less than or equal to  $x$ , the minimum gap. See Figure (1) and paragraph preceding Equation (3.1), in this paper and Figure (1) in Reference (11).



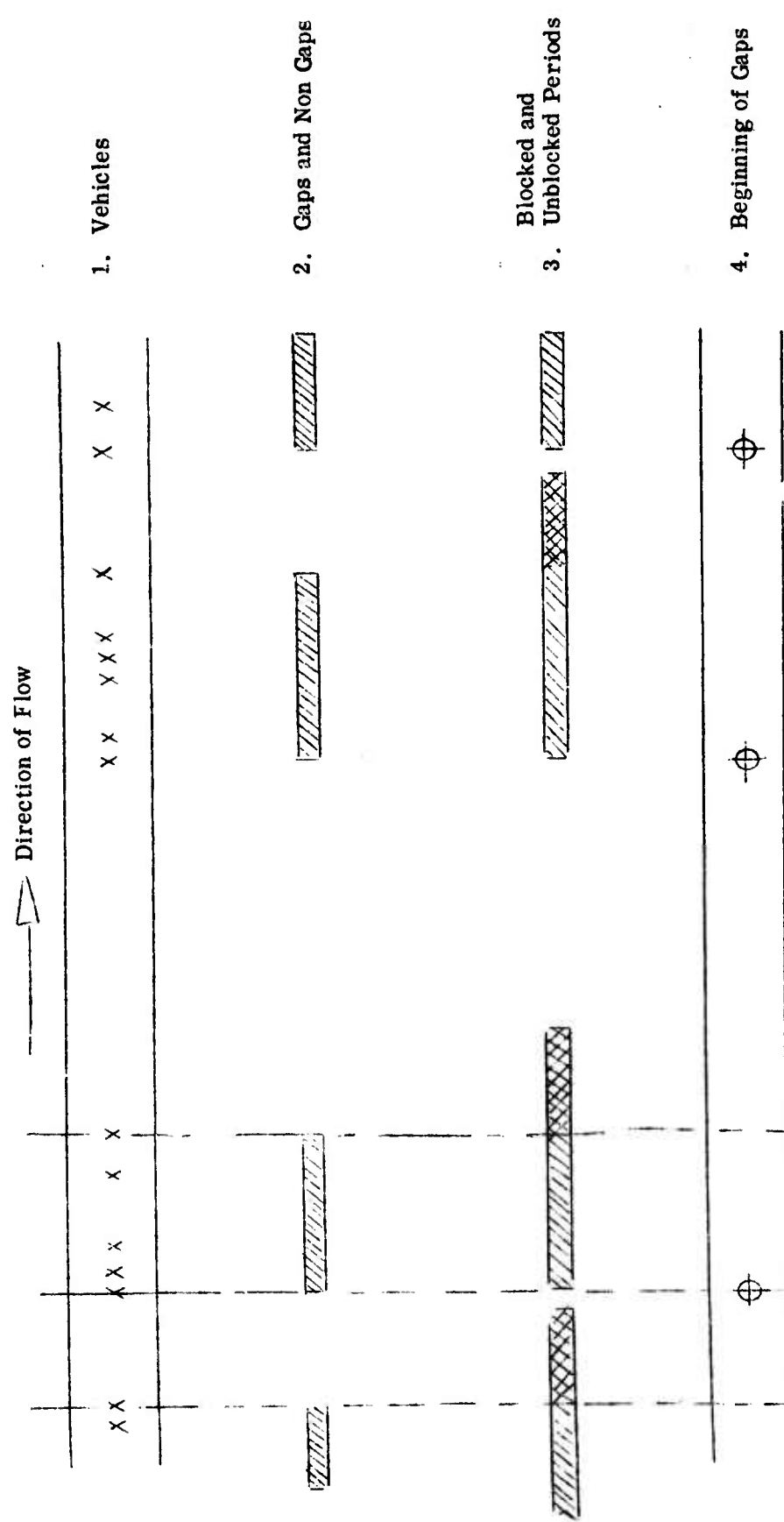


FIG. 1 - GAPS AND BLOCKS IN TRAFFIC STREAMS.

While the study of the gap-producing mechanism was generally incidental to the study of the probability distribution of wait for an unblocked period, several authors have studied the frequency of appearance and the distribution of large gaps in traffic streams. As early as 1936 Adams<sup>(1)</sup> obtained a formula for the average wait for an unblocked interval in a Poisson stream. In 1950 Garwood<sup>(4)</sup> calculated the probability distribution of delay of vehicles at intersections with a special type of actuated signal. In 1951 Raff<sup>(11)</sup> made use of these results to obtain the probability distribution of wait for an unblocked period and the probability distribution of block sizes. In their papers Garwood and Raff were only interested in Poisson traffic streams. The probability distribution of wait for an unblocked period was derived in three distinct ways by Tanner<sup>(13)</sup> when the traffic count in the major stream was Poisson. Mayne<sup>(7)</sup> obtained an integral equation for the conditional probability distribution of wait for an unblocked period given the spacing to the next vehicle in the traffic stream. He then derived the probability distribution of spacing between a randomly chosen time origin and the passing of the first vehicle and obtained the marginal distribution of wait in terms of its Laplace transform. Mayne<sup>(7)</sup> assumed that inter-vehicle spacings were arbitrary but independently distributed random variables. More recently, Weiss and Maradudin<sup>(14)</sup> have analyzed the problem with methods of renewal theory; they formulate the conditional probability distribution of spacings between vehicles in terms of the random variable which describes the wait of a vehicle in the minor stream. Special attention is given to a generalization of the waiting problem where vehicles in the minor stream are assigned a probability distribution of gap acceptances. Jewell<sup>(5)</sup> has derived the wait for an unblocked period for arbitrary arrival characteristics of the vehicle in the minor stream. He has also obtained results for the distribution of counts of vehicles crossing through unblocked periods when an infinite queue of vehicles appears in the minor stream. J. D. C. Little<sup>(6)</sup> has obtained approximate expressions for the delay of a vehicle in several distinct merging and crossing maneuvers.

Notation will be defined as it is used but for reference purposes it may be convenient to have a paragraph outlining the general structure of the probability distributions and the moments which measure inter-vehicle and inter-gap headways:

$a(t)$ : the probability density function that the headway between two vehicles is  $t$ .

$b(t)$ : the probability density function that the headway between a vehicle and the first vehicle which marks the beginning of a gap is  $t$ .

$c(t)$ : the probability density function that the headway from beginning of a gap to beginning of a gap is  $t$ .

The distribution of sizes of gaps and blocked periods are:

$f(t)$ : the probability density function of block sizes.

$g(t)$ : the probability density function of gap sizes.

$h(t)$ : the probability density function of sizes of unblocked periods.

If one starts to measure headways from a randomly located time origin,

$u(t)$ : the probability density function that the spacing to the first vehicle is  $t$ .

$v(t)$ : the probability density function that the spacing to the beginning of the first gap is  $t$ .

$w(t)$ : the probability density function that the spacing to the first unblocked period is  $t$ .

Cumulative (tail) distributions are indicated by a capital letter and the subscript  $n$  refers to a headway between  $n$  events, say  $n$  vehicles or  $n$  gaps. For example  $A_n(t)$  is the probability that the spacing from a vehicle to the  $n$ th vehicle is greater than or equal to  $t$ . It is understood that the absence of a numerical subscript refers to the  $n = 1$  case. The Laplace transform of a function is indicated by a tilde ( $\sim$ ) over the function and the dummy variable  $s$  in place of the transformed variate.

The average value of the headway between vehicles is indicated by  $\bar{v}$  and the variance by  $\sigma_v^2$ . Means and variances of the other distributions are distinguished by subscripts such as  $\bar{v}_b, \sigma_b^2; \bar{v}_w, \sigma_w^2 \dots$  etc. The stationary flow rate of vehicles is  $\mu$  and of gaps  $\mu_c$ .  $\delta(t)$  is the Dirac delta function.

# 1. The Count of Vehicles and Gaps

The first clues that gap statistics may be simple to derive appear when one neglects the length of the headway between successive vehicles. If one thinks of the passing of a vehicle as the occurrence of a trial which is either a success (i.e. followed by a gap) or a failure (i.e. no gap) there will be runs of gaps followed by runs of vehicles which do not begin gaps. One can assume that the probability of a gap equals the probability that the spacing between two vehicles is greater than or equal to  $x$ , i.e.  $A(x)$ . Since inter-vehicle spacings are identically but independently distributed the probability that for the first time the  $n$ th vehicle will result in a gap is the geometric distribution

$$p_n = A(x) (1 - A(x))^{n-1} \quad n \geq 1 \quad \dots \quad (1.1)$$

which, incidentally, counts the gap as a vehicle. The probability that a gap (the 1st, 2nd, ...  $n$ th) will occur with the  $n$ th vehicle is independent of the vehicle number and is just equal to the gap probability  $A(x)$ .

From this simple expression there is no difficulty in obtaining the binomial probability distribution  $p_{mn}$  of counting  $m$  gaps with the appearance of  $n \geq m$  vehicles

$$p_{mn} = \binom{n}{m} (A(x))^m (1 - A(x))^{n-m} \quad n \geq m \geq 1 \quad \dots \quad (1.2)$$

and the binomial distribution  $q_{mn}$  that there will be  $n$  vehicles up to and including the passing of the  $m$ th gap

$$q_{mn} = \binom{n-1}{m-1} (A(x))^m (1 - A(x))^{n-m} \quad n \geq m \quad \dots \quad (1.3)$$

Since a gap is counted by the vehicle which starts the gap, the  $m$ th gap must signal at least the passing of the  $m$ th vehicle. One counts  $m$  gaps and  $n-m$  vehicles but since a gap must end the sequence of trials the arrangement of the remaining  $m-1$  gaps is among  $n-1$  vehicles. If vehicles are distributed in a random Poisson fashion then the distribution of inter-vehicle spacings is exponential. If the average inter-vehicle separation is  $\mu^{-1}$  the probability that the spacing between vehicles is greater than or equal to a minimum gap  $x$  is  $e^{-\mu x}$ . Substituting  $e^{-\mu x}$  for  $\Lambda(x)$  in Equations (1.1)(1.2) one gets the Furry distribution,

$$p_n = e^{-\lambda x} \left(1 - e^{-\lambda x}\right)^{n-1} \quad n \geq 1 \quad \dots \quad (1.4)$$

for the probability that the first gap is begun by car  $n$  and

$$p_{mn} = \binom{n}{m} e^{-m\mu x} \left(1 - e^{-\mu x}\right)^{n-m} \quad n \geq m \geq 1 \quad \dots \quad (1.5)$$

for the discrete counting law of gaps.

The average number of vehicles which must go by before a gap is

$$\sum_{n=1}^{\infty} n p_n = \Lambda^{-1}(x) \quad \dots \quad (1.6)$$

i. e. just the reciprocal of the probability that the spacing between vehicles is greater

than or equal to  $x$ . It is tempting (and correct) to assume that the average headway,  $v_c$  from beginning of gap to beginning of gap is the product of the average inter-vehicle headway  $v$  and the average number of cars between gaps\*,

$$v_c = v A^{-1}(x) \dots \dots \dots (1.7)$$

By excluding the average length,  $v_g$ , of the gap itself one can obtain the average headway,  $v_b$ , from any vehicle to the beginning of a gap<sup>+</sup>.

$$v_b = v A^{-1}(x) - v_g \dots \dots \dots (1.8)$$

When inter-vehicle headways are exponentially distributed with mean  $\mu^{-1}$ , the average inter-gap headway is

$$v_c = \mu^{-1} e^{\mu x} \dots \dots \dots (1.9)$$

The reciprocal of the stationary gap flow rate  $\mu e^{-\mu x}$  and the average headway to the beginning of a gap is

\* See Equation (2.20)

+ See Equation (2.17)

$$v_b = \mu^{-1} e^{-\mu x} - \mu^{-1} - x \quad \dots \dots \dots (1.10)$$

a formula originally obtained by Adams<sup>(1)</sup> in 1936. While the special Poisson traffic streams leads to notable algebraic simplifications, it is conceptually no more difficult to consider arbitrary distributions for inter-vehicle headways. In order to calculate the headway between gaps or the count of gaps in a given interval one must include information about the headways between those vehicles which do not begin gaps as well as the lengths of gaps themselves.

## 2. The Distribution of Headways Between Vehicles and Gaps

The headway between every  $n$ th vehicle will consist of the sum of  $n$  inter-vehicle headways;  $m$  of these may be gaps and  $n-m$  may not be large enough to qualify as gaps. The joint probability  $a(t; x)dt$  that the headway between two vehicles lies between  $t$  and  $t + dt$  and that  $t$  is less than  $x$  has a density function,

$$\begin{aligned} a(t; x) &= a(t) & 0 \leq t < x & \dots \dots \dots (2.1) \\ &= 0 & x \leq t & \end{aligned}$$

where  $a(t)$  is the probability density of a headway  $t$  between adjacent vehicles. We note in passing that this is not the conditional probability distribution of headways given that headways are less than  $x$ . The probability that the headway between each of two successive vehicles is less than  $x$  and the sum of both headways lies between  $t$  and  $t + dt$  is  $a_2(t; x)dt$  where,

$$a_2(t; x) = \int_0^t a(r; x) a(t-r; x) dr \quad 0 \leq t < 2x \quad \dots \dots \dots (2.2)$$

In general, we find that the probability density distribution of headway between  $n$  vehicles where no gap is observed between any one vehicle is

$$a_n(t; x) = \int_0^t a_{n-1}(r; x) a(t-r; x) dr \quad 0 \leq t < nx \quad \dots \dots \dots (2.3)$$

and zero outside the interval  $[0, nx)$ .  $a_n(t; x)$  is the probability density of the sum of



$n$  inter-vehicle headways each less than  $x$ . The probability law which describes the headway between every  $n$ th vehicle when one does not distinguish between gaps and inter-vehicle spacings which are smaller than gaps is the  $n$ -fold convolution of the probability density of inter-vehicle headways,

$$a_n(t) = \int_0^t a_{n-1}(r) a(t-r) dr \quad \dots \dots \dots (2.4)$$

An observer of a traffic stream may find that the headway between the  $n$ th and  $(n+1)$ st vehicle is the first one greater than or equal to  $x$ . The probability that the headway between a randomly selected vehicle and this  $n$ th vehicle - i.e. the first to start a gap - lies between  $t$  and  $t+dt$  is

$$a_n(t; x) A(x) dt \quad n \geq 1 \quad \dots \dots \dots (2.5)$$

Since the first gap may appear with the passing of the first, second, ....  $n$ th vehicle the probability that the headway between a vehicle and the beginning of a gap greater than  $x$  lies between  $t$  and  $t+dt$  has a density function

$$b(t) = A(x) \delta(t) \quad t = 0 \quad \dots \dots \dots (2.6a)$$

$$= \sum_{n=1}^{\infty} a_n(t; x) A(x) \quad t > 0 \quad \dots \dots \dots (2.6b)$$

$b(t)$  is also the density distribution of "non-gap" sizes in Figure (1). To find the probability that the inter-gap headway, i.e. from beginning of gap to beginning of gap, is greater than or equal to  $t$ , we note that this headway is the sum of two random variables, one representing the total time from end of a gap to beginning of the succeeding gap, the other being the length of the gap itself. The conditional probability distribution of gap sizes is,

$$\begin{aligned} g(t) &= \dots \dots \dots & 0 \leq t < x \\ &= a(t) A^{-1}(x) & x \leq t \dots \dots \dots (2.7) \end{aligned}$$

and  $C(t)$ , the probability that the inter-gap headway is greater than or equal to  $t$ , can be represented as the convolution of this distribution with  $b(t)$  in Equation (2.6),

$$C(t) = \int_t^\infty \int_0^r b(r-y) g(y) dy dr \dots \dots \dots (2.8a)$$

$$= A^{-1}(x) \int_t^\infty \int_x^r b(r-y) a(y) dy dr \dots \dots \dots (2.8b)$$

Since we are for the most part considering convolution type integrals it is helpful to discuss Equations (2.1) through (2.8) in terms of their Laplace transforms. The Laplace transform of the inter-vehicle density distribution  $a(t)$  is

$$\tilde{a}(s) = \int_0^\infty a(t) e^{-st} dt \dots \dots \dots (2.9)$$

and of the joint probability distribution  $a(t; x)$

$$\begin{aligned}\tilde{a}(s; x) &= \int_0^{\infty} a(t; x) e^{-st} dt \\ &= \int_0^x a(t) e^{-st} dt \dots \dots \dots (2.10)\end{aligned}$$

Hence, the transform of the density distribution of headways between every  $n$  vehicles (Equation (2.4)) is

$$\tilde{a}_n(s) = \left( \tilde{a}(s) \right)^n \dots \dots \dots (2.11)$$

and between  $n$  vehicles when no gap appears between any two vehicles is

$$\tilde{a}_n(s; x) = \left( \tilde{a}(s; x) \right)^n \dots \dots \dots (2.12)$$

Substitution of Equation (2.12) into (2.6) and summation of the geometric series leads to the transform of the "vehicle-to-gap" distribution,

$$\tilde{b}(s) = A(x) \left( 1 - \tilde{a}(s; x) \right)^{-1} \dots \dots \dots (2.13)$$

Similarly, substitution of Equation (2.13) and (2.9) into (2.11) gives

$$\tilde{C}(s) = \frac{1}{s} \frac{1 - \tilde{a}(s)}{1 - \tilde{a}(s; x)} \dots\dots\dots (2.14)$$

the transform of the probability distribution of inter-gap headways. While any one of these transforms may be difficult to invert one can obtain simple expressions for the moments of these distributions.

The mean and variance of the headway between vehicles are obtained from the inter-vehicle distribution

$$v = \int_0^{\infty} t a(t) dt \dots\dots\dots (2.15a)$$

$$\sigma^2 = \int_0^{\infty} (t - v)^2 a(t) dt \dots\dots\dots (2.15b)$$

The mean and variance of the gap sizes are

$$v_g = \int_0^{\infty} t g(t) dt \dots\dots\dots (2.16a)$$

$$\sigma_g^2 = \int_0^{\infty} (t - v_g)^2 g(t) dt \dots\dots\dots (2.16b)$$

The average vehicle-to-gap headway can be obtained by evaluating the negative derivative of  $\tilde{b}(s)$  at  $s = 0$ ,

$$v_b = v A^{-1}(x) - v_g \quad \dots \dots \dots (2.17)$$

If we recall Equation (1.7) we note that the vehicle to gap headway is the difference between the average inter-gap headway and the average gap size in Equation (2.16a).  $v_b$  can also be written as

$$v_b = \int_0^x t a(t) A^{-1}(x) dt \quad \dots \dots \dots (2.18)$$

which equals the average number,  $\frac{1 - A(x)}{A(x)}$ , times the average size of those inter-vehicle headways which do not qualify as gaps. \*

The variance of the vehicle to gap distribution  $b(t)$  is obtained from the first and second derivatives of Equation (2.13)

$$\begin{aligned} \sigma_b^2 &= v_b^2 + \int_0^x t^2 a(t) A^{-1}(x) dt \quad \dots \dots \dots (2.19) \\ &= v_b^2 + \frac{\sigma^2 + v^2}{A(x)} - (\sigma_g^2 + v_g^2) \end{aligned}$$

The average inter-gap headway obtained by evaluating  $\tilde{C}(s)$  at  $s = 0$ ,

$$v_c = v A^{-1}(x) \quad \dots \dots \dots (2.20)$$

\*From Equation (1.6) we note that  $\frac{1 - A(x)}{A(x)}$  is just the average number of vehicles between but not including the ones which begin gaps.

is the result already obtained in Equation (1.7). The variance of this headway is obtained in terms of the value of the derivative of  $\tilde{C}(s)$  at  $s = 0$ ,

$$\sigma_c^2 = A^{-1}(x) \left\{ \sigma^2 - v^2 (1 - A(x)) A^{-1}(x) + 2vv_b \right\} \dots \dots \dots (2.21)$$

$\sigma_c^2$  can also be found with the help of the variance theorem for sums of independently distributed random variables.

$$\begin{aligned} \sigma_c^2 &= \sigma_b^2 + \sigma_g^2 \\ &= A^{-1}(x) \left\{ \sigma^2 + v^2 (1 + A(x)) A^{-1}(x) - 2vv_g \right\} \dots \dots \dots (2.22) \end{aligned}$$

Substituting Equation (2.17) into (2.22) leads to Equation (2.21). In other words Equations (2.21) and (2.22) are equivalent expressions for the variance of the inter-gap headways. While the former is a function of the mean vehicle to gap headway the latter is a function of the mean gap size.

So far we have discussed time origins which start with the passing of a randomly chosen vehicle or those vehicles which begin a gap. We will make use of still another time origin in our study of the flow of gaps in the traffic stream. If we start at random to measure a headway to the beginning of a gap we will refer to its probability density function  $v(t)$ . It is well known from renewal theory <sup>(12)</sup> that

$$v(t) = v_c^{-1} C(t) \quad \dots \dots \dots (2.23)$$

which has a transform equal to  $v_c^{-1}$  times the transform of Equation (2.14). One can also derive Equation (2.23) by a second line of argument: If one starts to observe traffic at random the probability that the headway to the first vehicle lies between  $t$  and  $t+dt$  is  $u(t) = v^{-1} A(t)$ ; the probability density distribution of headway from this vehicle to the beginning of the first gap is  $b(t)$  in Equation (2.6). The convolution of these two probability distributions is

$$v(t) = \int_0^t u(r) b(t-r) dr \quad \dots \dots \dots (2.24)$$

with transform

$$\tilde{v}(s) = \frac{A(x)}{v s} \frac{1 - \tilde{a}(s)}{1 - \tilde{a}(s; x)} \quad \dots \dots \dots (2.25)$$

which is also the transform of Equation (2.23). As one might expect the statement "starting at random" can refer to either the passing of a vehicle or the passing of a gap.

The average value of the distribution  $v(t)$  is the average wait for the beginning of the first gap if an observer arrives at random. It can be expressed as the sum of the average headway to the first vehicle,

$$v_u = \int_0^{\infty} t u(t) dt = \frac{\sigma^2 + v^2}{2v} \dots \dots \dots (2.26)$$

and the average headway  $v_b$  from the first vehicle to the first gap. Substitution of  $v_c$  for  $v$  and  $\sigma_c^2$  for  $\sigma^2$  in Equation (2.26) leads to the expression

$$\begin{aligned} v_v &= \frac{\sigma_c^2 + v_c^2}{2v_c} = v_u + v_b \\ &= \frac{\sigma^2}{2v} + v \left( \frac{2 + A(x)}{2A(x)} \right) - v_g \dots \dots \dots (2.27) \end{aligned}$$

in terms of the moments of the inter-vehicle distribution and the average gap size  $v_g$ .



### 3. Block Sizes and the Wait for Unblocked Periods

Until this point we have only been concerned with the statistical properties of the intervals between gaps and the intervals from selected points (vehicles, random origins, end of gaps, etc.) to the beginning of a gap. In earlier papers some of these results were derived in terms of the wait of a minor stream vehicle which looks for the first unblocked period. We will denote the probability density function of wait for an unblocked period by  $w(t)$  and point out that this is not the same function as  $v(t)$  in Equation (2.23). This distinction is made clear by considering the differences between blocked periods, unblocked periods and gaps in Figure (1). A blocked interval is one which begins a time  $x$  prior to the end of a gap and terminates with the beginning of the next gap greater than  $x$ . In the third line of Figure (1) we notice that a vehicle which arrives during a blocked (shaded) period must also wait for the first gap but one which arrives in an unblocked period does not have to wait at all. The major difference between the distribution of wait for an unblocked period and the wait for the beginning of the next gap is the term expressing zero wait; it is, of course, non-zero in the former case. With low flow rates in the major stream the probability that a random arrival in the minor stream (i.e. the measurement origin) is located in the middle of an unblocked period is large. When flow rates are large the probability of zero wait should be small and  $w(t)$  should resemble  $v(t)$ .

In the introduction I made reference to the fact that the density distribution,  $w(t)$  had been calculated by several authors. By considering the sequence of events which gives rise to no wait or to a positive wait it is possible to obtain  $w(t)$  in terms of the probability density,  $b(t)$ , of headways from a vehicle to the beginning of a gap. The probability of a zero wait for an unblocked period is the probability that the randomly chosen time origin is separated by more than  $x$  from the first vehicle, i.e.  $U(x)$ . If

we define distributions  $u(t; x)$  and  $u_n(t; x)$  (similar to  $a(t; x)$  and  $a_n(t; x)$  in Equations (2.1)(2.3)) which are the probability density distributions of headways from a random origin to the first or  $n$ th vehicle, we can find the probability of positive wait for an unblocked period. This wait is the sum of the wait to the first vehicle (which must arrive before  $x$ ) and the wait from the first vehicle to the beginning of the first gap.

$$\begin{aligned} w(t) &= U(x) \delta(t) & t &= 0 \\ & & & \dots \dots \dots (3.1) \\ &= \int_0^t u(r; x) b(t-r) dr & t &> 0 \end{aligned}$$

with transform\* equal to

$$\tilde{w}(s) = U(x) + \frac{A(x) \tilde{u}(s; x)}{1 - \tilde{a}(s; x)} \dots \dots \dots (3.2)$$

This result was obtained in a slightly different form by Mayne<sup>(7)</sup>.  $v_w$ , the average wait, is smaller than  $v_v$  since (i) the probability of zero wait for an unblocked period is positive, (ii) a positive wait implies a wait for the beginning of a gap.

$$v_w = (1 - U(x))v_b + \int_0^x t u(t) dt \dots \dots \dots (3.3)$$

\* Jewell<sup>(5)</sup> obtained Equation (3.2) for more general "starting-up" distributions, i.e., if the distribution of headways to the first vehicle is  $d(t)$  he shows that,

$$\tilde{w}(s) = D(x) + A(x) \tilde{d}(s; x) (1 - \tilde{a}(s; x))^{-1}$$

Note that when  $d(t) = a(t)$  one gets  $\tilde{w}(s) = \tilde{b}(s)$ .

$$v_w = v_v - U(x)v_b - \frac{1}{2v_c} \left[ \sigma_g^2 + v_g^2 - x^2 \right]$$

The distribution of block sizes is itself an interesting one since the calculations enables us to generalize on some results obtained by Raff<sup>(11)</sup>.

Let  $f(t)$  be the density distribution of block sizes and  $h(t)$  be the density distribution of lengths of unblocked periods. Blocks can be constructed from "non-gaps" by extending the latter by  $x$ . Hence, the distribution  $f(t)$  of block sizes is the translated distribution of  $b(t)$ ,

$$f(t) = b(t - x) \quad t \geq x \quad \dots \dots \dots (3.4)$$

and is zero for values of headway less than the minimum gap. The distribution  $h(t)$  of sizes of unblocked intervals is the solution of that convolution (with  $f(t)$ ) which leads to the distribution of spacings from beginning of an unblocked period to the beginning of a unblocked period. Since an unblocked period is always followed by a blocked period and since the beginning of an unblocked period is also the beginning of a gap the convolution mentioned above must also equal the probability distribution of inter-gap headways,

$$\begin{aligned} c(t) &= 0 & 0 \leq t < x \\ &= \int_0^t f(r) h(t-r) dr & t \geq x \quad \dots \dots \dots (3.5) \end{aligned}$$

By direct geometrical argument from Figure (1) one can see that a gap of length  $t + x$  corresponds to an unblocked interval of length  $t$ . Hence, the solution of  $h(t)$  in Equation (3.5) is the translated distribution of gap sizes in Equation (2.7)

$$h(t) = g(t+x) \quad t \geq 0 \quad \dots \dots \dots (3.6)$$

$$= \Lambda^{-1}(x) \left\{ a(t+x) - a(t+x; x) \right\}$$

The product of the transform of the block size distribution

$$\tilde{f}(s) = e^{-sx} \tilde{b}(s) \quad \dots \dots \dots (3.7a)$$

and the distribution of lengths of unblocked periods

$$\tilde{h}(s) = \Lambda^{-1}(x) e^{sx} \left[ \tilde{a}(s) - \tilde{a}(s; x) \right] \quad \dots \dots \dots (3.7b)$$

must equal the transform of the inter-gap distribution. From Equations (2.13)(2.14) and (3.7)

$$\tilde{f}(s) \tilde{h}(s) = \frac{\tilde{a}(s) - \tilde{a}(s; x)}{1 - \tilde{a}(s; x)} = \tilde{c}(s) \quad \dots \dots \dots (3.8)$$

The average and variance of the blocked and unblocked periods are

$$v_f = v_b + x = v \Lambda^{-1}(x) - v_g + x \quad \dots \dots \dots (3.9a)$$

$$\sigma_f^2 = \sigma_b^2 \quad \dots \dots \dots (3.9b)$$

$$v_h = v_g - x \quad \dots \dots \dots (3.9c)$$

$$\sigma_h^2 = \sigma_g^2 \quad \dots \dots \dots (3.9d)$$

There is no difficulty in finding expressions for the wait,  $w(t)$ , for an unblocked period in terms of the distribution of block sizes. The probability of zero wait is, as before,  $U(x)$ , the probability that the minor stream vehicle arrives during an unblocked period. The probability that the wait is  $t > 0$  is the product of (i) the probability that the wait is not zero and (ii) the probability that the interval from the randomly located time origin to the end of a block is  $t$ , i.e.

$$w(t) = (1 - U(x)) v_f^{-1} F(t) \quad t > 0 \quad \dots \dots \dots (3.10)$$

The transform for  $w(t)$  includes the probability of zero wait,

$$\tilde{w}(s) = U(x) + (1 - U(x)) v_f^{-1} \tilde{F}(s) \quad \dots \dots \dots (3.11)$$

Equation (3.10) is an interesting one since it generalizes on a result obtained by Raff<sup>(11)</sup> for Poisson traffic.  $w(t)$  is always a linear function of  $F(t)$  but  $w(t)$  is only a linear function of  $W(t - x)$  when the traffic is Poisson. (See remarks following Equation (5.17)).

In order to show that Equations (3.2) and (3.11) are equivalent expressions, it is helpful to state and prove two identities, the second of these related to the probability

of zero wait for an unblocked period and a number which Weiss and Maradudin<sup>(14)</sup> have chosen to call the transparency of a stream of traffic - i.e., the fraction of time the major stream is not blocked.

The first one,

$$\tilde{u}(s; x) = \frac{1}{vs} \left\{ 1 - \tilde{a}(s; x) - A(x) e^{-sx} \right\} \dots \dots \dots (3.12)$$

is easily obtained by partial integration of the transform of

$$u(t; x) = v^{-1} \Lambda(t; x) \dots \dots \dots (3.13)$$

The relation between the probability of zero wait for an unblocked period and the average length of those periods,

$$v_h v_c^{-1} = U(x) \dots \dots \dots (3.14)$$

appears repeatedly in a discussion of gaps, blocked or unblocked periods\*.

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\* Weiss and Maradudin<sup>(14)</sup> have shown that with their gap acceptance criteria the transparency may not equal  $U(x)$ , the probability of zero wait.

Equation (3.14) can be proved in several ways. A direct and simple proof makes use of Equations (2.20), (3.9) and the partial integration of  $U(x)$ ,

$$U(x) = v^{-1} \int_x^{\infty} A(t) dt \quad \dots \dots \dots (3.15)$$

$$= A(x) v^{-1} v_g - x v_c^{-1} = v_h v_c^{-1}$$

By substituting  $\frac{1}{s} \left( 1 - e^{-sx} \tilde{b}(s) \right)$  for  $\tilde{F}(s)$ ,  $v_h v_c^{-1}$  for  $U(x)$  (Equation (3.14)) and Equation (3.12) into (3.11) one obtains the original expression for the transform of wait (Equation (3.2))

$$\tilde{w}(s) = U(x) + \frac{A(x) \tilde{u}(s; x)}{1 - \tilde{a}(s; x)} \quad \dots \dots \dots (3.16)$$

in terms of the inter-vehicle statistics. With Equation (3.14) one can also obtain an expression for the average wait,  $v_w$ , in terms of the second moment of the length of blocked intervals. From Equation (3.11) one obtains

$$\frac{d\tilde{w}}{ds} = \left( 1 - U(x) \right) v_f^{-1} \frac{d\tilde{F}}{ds} \quad \dots \dots \dots (3.17a)$$

and an equivalent form of Equation (3.3),

$$v_w = \frac{\sigma_f^2 + v_f^2}{2 v_c} \quad \dots \dots \dots (3.17b)$$

It follows that  $v_w$  is always less than  $v_v$ .

#### 4. Stationary Vehicle and Gap Flow Rates

Once the distribution of inter-gap headways is known it is in principle easy to obtain the discrete counting laws for gaps and blocks. Since there is a one to one correspondence between gaps and unblocked periods it will suffice to find counting distributions for the number of gaps in an interval. Both gaps and blocked periods have a minimum size of  $x$ , hence, the maximum number which can be counted in an interval  $t$  is the integral part of  $tx^{-1}$  or  $N = \left[ tx^{-1} \right]$ . The probability that exactly  $N$  gaps are counted in  $t$  may be large.

The probability  $p_n(t)$  that  $n \leq N^*$  gaps are counted in  $t$  if one starts to count with the passing of a gap is the convolution of the probability that the spacing to the first gap is  $r$  and  $n-1$  gaps appear in  $t-r$ ,

$$p_0(t) = C(t) \quad \dots \dots \dots (4.1a)$$

$$p_n(t) = \int_0^t c(r) p_{n-1}(t-r) dr \quad 1 \leq n < N \quad \dots \dots \dots (4.1b)$$

with transform

$$\tilde{p}_n(s) = \tilde{C}(s) \left[ 1 - s \tilde{C}(s) \right]^n \quad n \geq 0 \quad \dots \dots \dots (4.2)$$

The transform of the average count of gaps in an interval is,

$$\tilde{n}(s) = \sum_{n=0}^{\infty} n \tilde{p}_n(s) = \frac{\tilde{a}(s) - \tilde{a}(s; x)}{s(1 - \tilde{a}(s))} \quad \dots \dots \dots (4.3)$$

\* The first gap is not counted and  $1 \leq n \leq N$ .



and the stationary flow rate of gaps (i. e. , gap count per unit time) is

$$\begin{aligned}\mu_c &= \lim_{t \rightarrow \infty} \frac{1}{t} \sum_{n=0}^N n p_n(t) \\ &= \lim_{s \rightarrow 0} s \int_0^\infty \tilde{n}(s) ds = v_c^{-1} \dots \dots \dots (4.4)\end{aligned}$$

The limiting expression for the variance of this gap flow rate is<sup>(12)</sup>

$$\begin{aligned}\eta_c &= \lim_{t \rightarrow \infty} \frac{1}{t} \sum_{n=0}^N (n - \mu_c)^2 p_n(t) \\ &= \sigma_c^2 \mu_c^3 \dots \dots \dots (4.5)\end{aligned}$$

From Equations (2.20) and (2.22) we obtain the mean and variance of the gap flow rate in terms of the mean  $\mu = v^{-1}$  and variance,  $\eta = \sigma^2 \mu^3$  of the stationary vehicle flow rate

$$\mu_c = \mu A(x) \dots \dots \dots (4.6a)$$

$$\eta_c = \eta A^2(x) + \mu_c \left( 1 + A(x) \right) - 2\mu_c^2 v_g \dots \dots \dots (4.6b)$$

The first expression becomes obvious by its simplicity for it is just the stationary vehicle flow rate,  $\mu$ , multiplied by the probability that any two vehicles are separated by a gap.

### 5. Poisson Flows

A case of special interest occurs when the vehicles arrive at random, i.e., vehicle counts are Poisson. The distribution of inter-vehicle headways is exponential with mean  $\mu^{-1}$  and the transforms of Equations (2.9) and (2.10) are

$$\tilde{a}(s) = \tilde{u}(s) = \frac{\mu}{\mu + s} \dots \dots \dots (5.1)$$

$$\tilde{a}(s; x) = \tilde{u}(s; x) = \frac{\mu}{\mu + s} \left( 1 - e^{-(\mu + s)x} \right) \dots \dots \dots (5.2)$$

Since the distribution of headway from any time origin to the first vehicle is exponential a random time origin can be regarded as the passing of a vehicle. Hence, the distribution of wait for the first unblocked period and the distribution of vehicle to gap headway are identical. Equations (2.13) and (3.2) become

$$\tilde{b}(s) = \tilde{w}(s) = \frac{(\mu + s)e^{-\mu x}}{s + \mu e^{-(\mu + s)x}} \dots \dots \dots (5.3)$$

with an inverse equal to the density distribution of wait,

$$b(t) = w(t) = e^{-\mu x} \delta(t) \quad t = 0 \dots \dots \dots (5.4)$$

$$= \sum_{j=0}^N (-1)^j \mu^{j+1} e^{-(j+1)\mu x} \frac{(t-jx)^j}{j!} + \sum_{j=1}^N (-\mu)^j e^{-(j+1)\mu x} \frac{(t-jx)^{j-1}}{(j-1)!}$$

$$t > 0$$

where  $N = \lceil tx^{-1} \rceil$ . This expression was originally obtained and plotted by Raff<sup>(11)</sup> and Tanner<sup>(13)</sup> in terms of the cumulative distribution.

$$B(t) = W(t) = \sum_{i=0}^{j-1} (-1)^i e^{-(i+1)\mu x} \left\{ \frac{[\mu(t-ix)]^i}{i!} + \frac{[\mu(t-ix)]^{i+1}}{(i+1)!} \right\} \dots (5.5)$$

$$(j-1)x \leq t \leq jx$$

The average wait for an unblocked period

$$v_w = v_b = \mu^{-1} e^{\mu x} - x - \mu^{-1} \dots (5.6)$$

equals the expression we obtained in Equation (1.10) for the average vehicle to gap headway. The expression for the variance of wait

$$\sigma_w^2 = \sigma_b^2 = \frac{e^{2\mu x} - 2\mu x e^{\mu x} - 1}{\mu^2} \dots (5.7)$$

was originally obtained by Tanner<sup>(13)</sup>.

For small gaps the average wait for an unblocked period increases quadratically with  $x$  and linearly with the flow rate of vehicles\*,

\*Jewell<sup>(5)</sup> shows that Equation (5.8) is valid for arbitrary inter-vehicle distributions if we interpret  $\mu$  as the leading term in the power series expansion of

$$d(t) = \sum_{i=0}^{\infty} \frac{\mu^i t^i}{i!} \quad \text{in the footnote following Equation (3.1).}$$

$$v_b = v_w = \frac{\mu x^2}{2} + O(\mu^2 x^3) \dots \dots \dots (5.8)$$

The transform of the probability that the inter-gap headway is greater than  $t$  is

$$\tilde{C}(s) = \left[ s + \mu e^{-(\mu+s)x} \right]^{-1} \dots \dots \dots (5.9)$$

with inverse

$$C(t) = \sum_{j=0}^{\infty} \frac{[\mu(jx - t)]^j}{j!} e^{-j\mu x} \dots \dots \dots (5.10)$$

Figure (2) is a plot of  $C(t)$  as a function of  $t$  for several values of vehicle flow rates  $\mu$ .

The average inter-gap headway is

$$v_c = \mu^{-1} e^{\mu x} \dots \dots \dots (5.11)$$

For small gaps this average headway approaches the average inter-vehicle headway plus the length of the minimum gap. The variance of the inter-gap distribution is

$$\sigma_c^2 = \frac{e^{\mu x}}{\mu^2} (e^{\mu x} - 2\mu x) \dots \dots \dots (5.12)$$

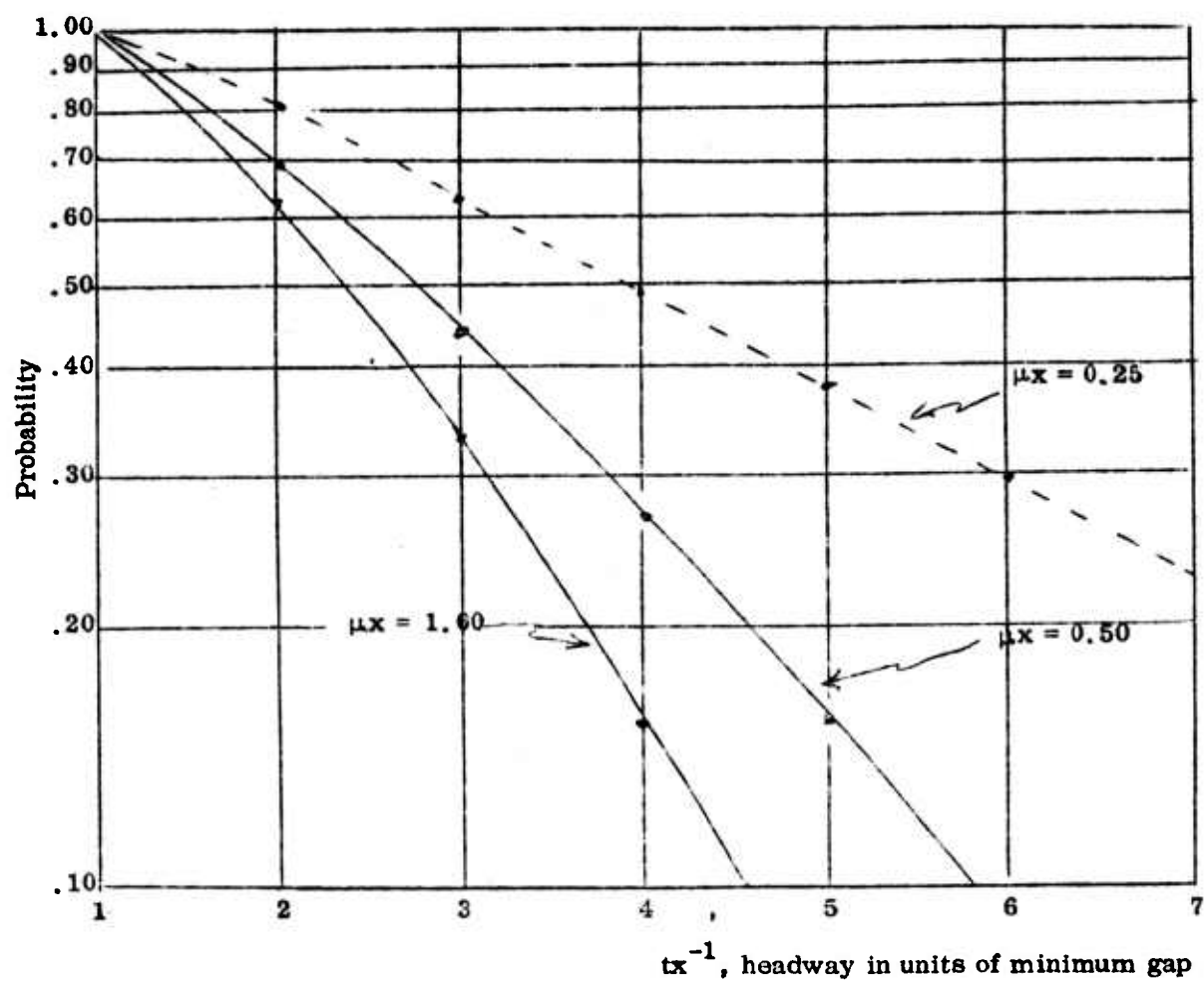


FIGURE 2 — THE DISTRIBUTION,  $C(t)$ , OF HEADWAY BETWEEN GAPS

which for small  $x$  is cubic in  $x$ ,

$$\sigma_c^2 = \frac{1}{\mu^2} + \frac{2}{3}\mu x^3 + \mathcal{O}(x) \quad \dots \dots \dots (5.13)$$

The probability density distribution of headways from a randomly chosen time origin is  $\mu e^{-\mu x}$  times  $C(t)$  in Equation (5.10) with average headway

$$v_v = \mu^{-1} e^{\mu x} - x \quad \dots \dots \dots (5.14)$$

The density distribution of gap sizes is a shifted exponential,

$$g(t) = \mu e^{-\mu(t-x)} \quad t \geq x \quad \dots \dots \dots (5.15)$$

with average gap size equal to the minimum gap plus the average inter-vehicle spacing,

$$v_g = x + \mu^{-1} \quad \dots \dots \dots (5.16a)$$

and variance

$$\sigma_g^2 = \mu^{-2} \quad \dots \dots \dots (5.16b)$$

As we saw in Equation (3.4) the distribution of block sizes is equal to the shifted distribution of vehicle to gap headways. Since, for Poisson traffic,  $b(t) = w(t)$  we also obtain

$$F(t) = W(t - x) \quad t \geq x \quad \dots \dots \dots (5.17)$$

$$= \sum_{i=1}^j (-1)^{i-1} e^{-ix} \left\{ \frac{[\mu(t-ix)]^{i-1}}{(i-1)!} + \frac{[\mu(t-ix)]^i}{i!} \right\}$$

$$jx \leq t \leq (j+1)x$$

a result which was used by Raff<sup>(11)</sup> to derive the distribution of block sizes from the distribution of wait for an unblocked period. Clearly, this result and the statement that the density function  $w(t)$  is a linear function of  $W(t - x)$  does not extend to non-Poisson traffic streams. (See Equation (3.11)). The distribution of lengths of unblocked periods is also exponential with mean  $\mu^{-1}$ .

Finally, we obtain the transform of the average count in  $t$  from Equation (4.3)

$$\tilde{n}(s) = \frac{1}{s^2} e^{-(\mu+s)x} \quad \dots \dots \dots (5.18)$$

with inverse

$$\bar{n}(t) = \mu(t - x) e^{-\mu x} \quad t \geq x \quad \dots \dots \dots (5.19)$$

The average and variance of the stationary gap flow rate are

$$\mu_c = \mu e^{-\mu x} \dots \dots \dots (5.20a)$$

$$\eta_c = \mu e^{-\mu x} - 2 \mu^2 x e^{-2\mu x} \dots \dots \dots (5.20b)$$

The gap flow rate  $\mu_c$  is plotted in Figure (4) as a function of the vehicle flow rate. When the vehicle flow rate is small the gaps are large, but they appear infrequently. As the vehicle flow rate increases the flow rate of gaps increases, reaches a maximum when  $\mu = x^{-1}$  and falls off rapidly as the probability of large headways between adjacent vehicles becomes small.

Few references have been made to experimental work in this area. For the sake of a large number of readers who may not have access to the paper by Moskowitz<sup>(9)</sup>, I am reprinting one of his many experimental curves\*. In Figure (3) Moskowitz has plotted the probability that the wait for an unblocked period is greater than or equal to  $t$  for several vehicle flow rates. The experimental and theoretical curves show very close fits for low and medium flow rates in the major stream. For an accurate description of the counting experiment and more detailed results the reader is referred to the original paper.

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\* With the kind permission of the author.



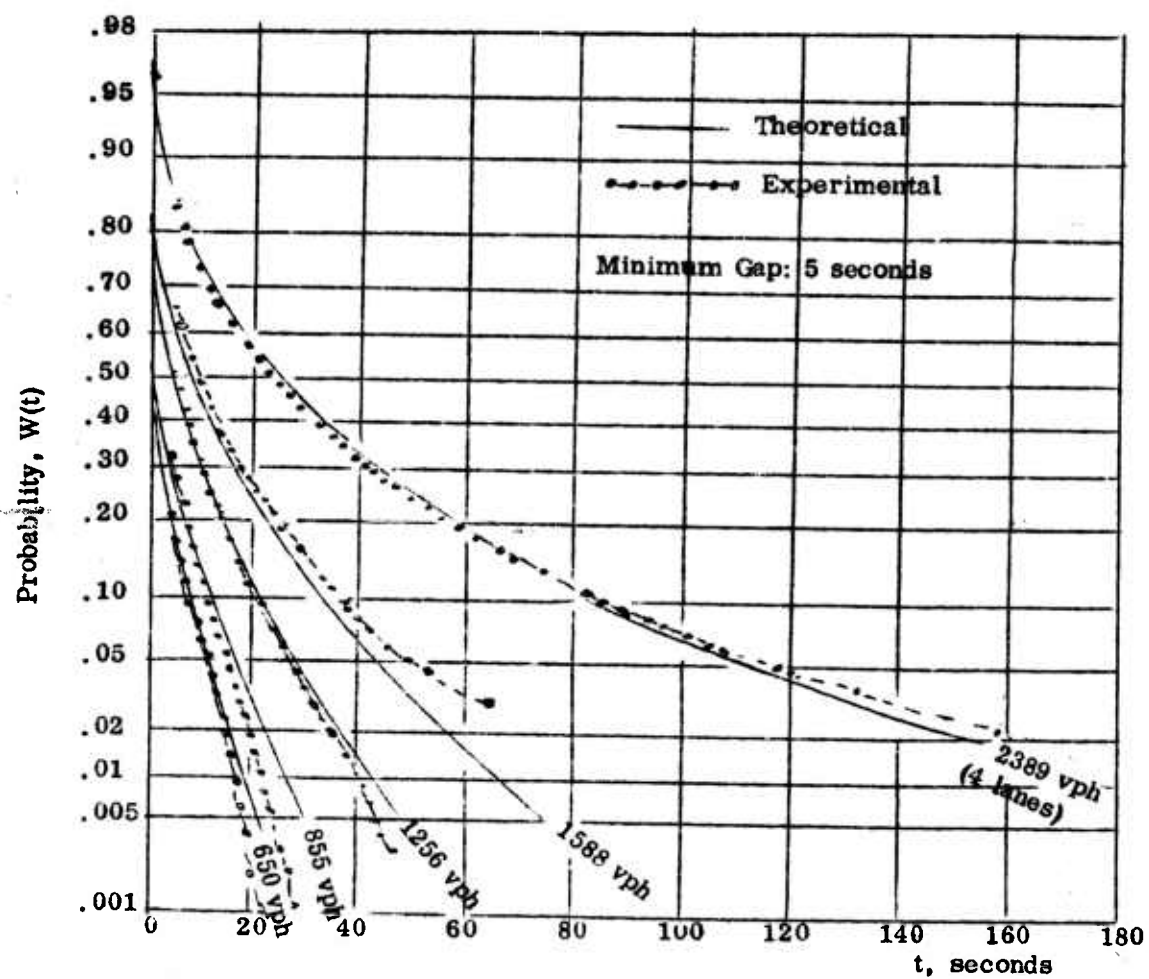


FIGURE 3 — THE PROBABILITY OF WAIT FOR AN UNBLOCKED PERIOD  
(Moskowitz Data: Reference (9))

## 6. Geometric Bunching With A Minimum Headway

Weiss and Maradudin<sup>(14)</sup> have pointed out that a generalization of the translated exponential distribution

$$A(t) = 1 \quad 0 \leq t \leq \Delta \quad \dots \dots \dots (6.1)$$

$$= e^{-\mu(t-\Delta)} \quad \Delta < t$$

is the probability distribution which allows geometric bunching (for example, runs of "fast" cars behind "slow" cars) with a minimum headway  $\Delta$  behind each vehicle,

$$A(t) = 1 \quad 0 \leq t \leq \Delta \quad \dots \dots \dots (6.2)$$

$$= (1-\alpha) e^{-\beta(t-\Delta)} \quad \Delta < t$$

The average value and variance of the vehicle to gap, inter-gap, gap and block size distributions are summarized in Table (1). Equations are expressed in terms of  $\mu$ , the vehicle flow rate,  $\alpha$  the fraction of vehicles in bunches, and  $\Delta$ , the minimum headway. The stationary gap flow rate,  $\mu_c = \mu (1-\alpha) e^{\frac{-\mu(1-\alpha)(x-\Delta)}{1-\mu\Delta}}$ , is plotted in Figure (4) in addition to the gap flow rate in Poisson traffic streams. The minimum gap size corresponds to a 5 second headway. With the probability distribution of Equation (6.2) which includes a minimum inter-vehicle separation,  $\Delta$ , the gap flow rate decreases more rapidly with large vehicle flow rates than for the Poisson case. In fact, the average vehicle flow rate cannot exceed  $\Delta^{-1}$ .

TABLE - 1

| Distribution           | Average  | Variance  |
|------------------------|--|---|
| Inter-vehicle: a(t)    | $v = \frac{1}{\mu}$  | $\sigma^2 = \frac{1+\alpha}{1-\alpha} \left( \frac{1-\mu\Delta}{\mu} \right)^2$                             |
| Vehicle-to-gap: b(t)   | $v_b = \frac{e^y}{\mu(1-\alpha)} - \frac{(1-\mu\Delta)}{\mu(1-\alpha)} - x$          | $\sigma_b^2 = \frac{e^{2y}}{\mu^2(1-\alpha)^2} + ze^y - \left( \frac{1-\mu\Delta}{\mu(1-\alpha)} \right)^2$ |
| Inter-gap: c(t)        | $v_c = \frac{e^y}{\mu(1-\alpha)}$  | $\sigma_c^2 = \frac{e^{2y}}{\mu^2(1-\alpha)^2} + ze^y$  |
| Block size: f(t)       | $v_f = \frac{e^y}{\mu(1-\alpha)} - \left( \frac{1-\mu\Delta}{\mu(1-\alpha)} \right)$ | $\sigma_f^2 = \sigma_b^2$   |
| Gap size: g(t)         | $v_g = \frac{1-\mu\Delta}{\mu(1-\alpha)} + x$  | $\sigma_g^2 = \left( \frac{1-\mu\Delta}{\mu(1-\alpha)} \right)^2$   |
| Unblocked Period: h(t) | $v_h = \frac{(1-\mu\Delta)}{\mu(1-\alpha)}$  | $\sigma_h^2 = \sigma_g^2$   |

NOTE:

$$y = \frac{\mu(1-\alpha)(x-\Delta)}{1-\mu\Delta} ; \quad z = \frac{\Delta^2 - 2x\Delta}{1-\alpha} - \frac{2x}{\mu} \left( \frac{1-\mu\Delta}{1-\alpha} \right) - \frac{2\Delta\alpha}{\mu} \frac{1-\mu\Delta}{(1-\alpha)^2}$$

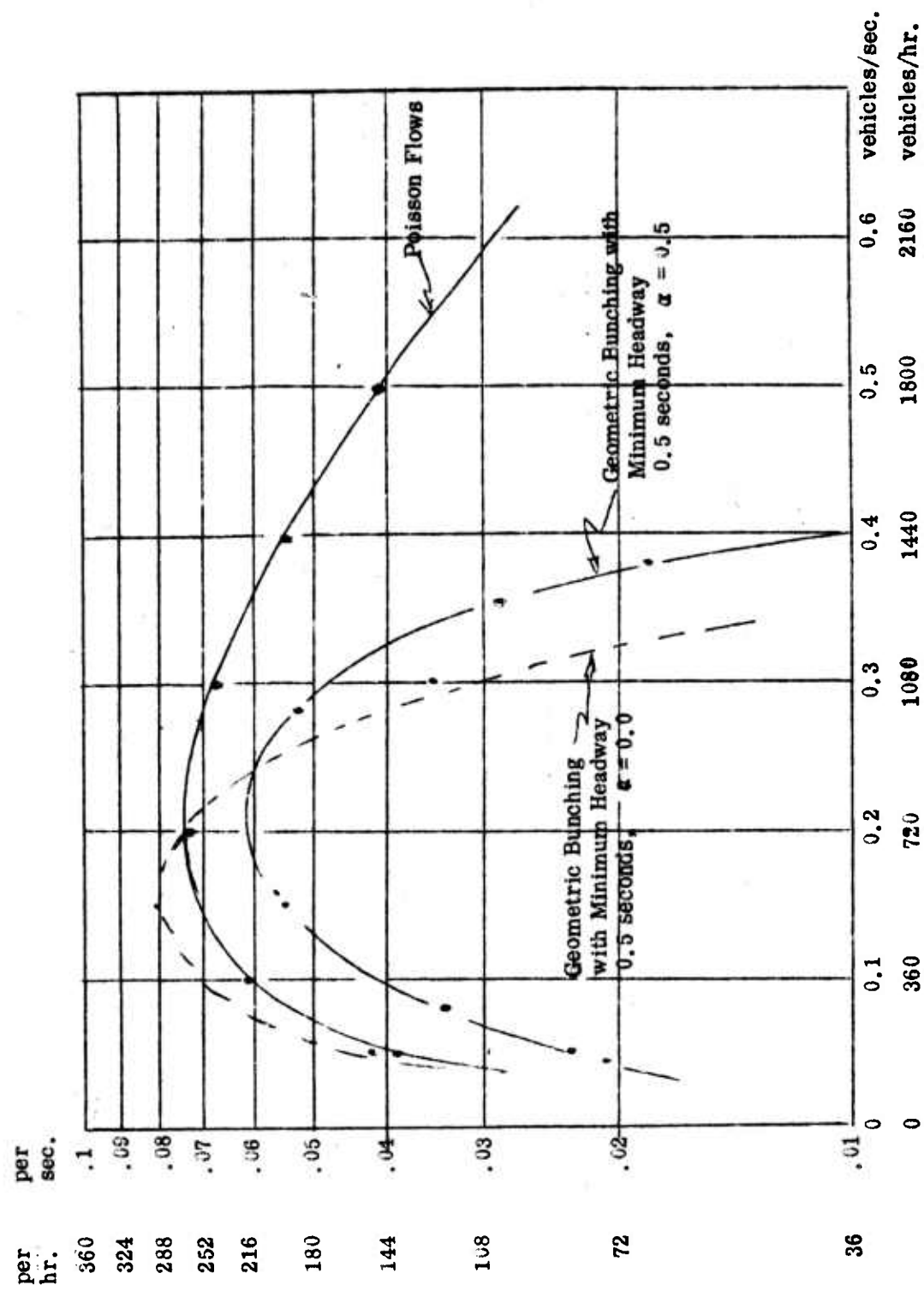


FIG. 4 - AVERAGE GAP AND BLOCK FLOW RATE  
(Minimum Gap: 5 seconds)

The effect of bunching is invariably that of giving smaller gap flows at low vehicle flows and higher gap flows at high vehicle flows than in the 'no-bunching' case. Bunching represents a concentration of vehicles at the minimum headway; for the same average inter-vehicle headway (i.e., the same vehicle flow rate) the probability of big headways is also larger than in the no-bunching case. At high vehicle flow rates the gap flow rate is sensitive to the tail of the inter-vehicle headway distribution; hence, the slower decrease in the gap flow rate.

## 7. Summary

As I mentioned in the introduction, I feel that there are at least two important ways of looking at the statistical flow processes which generate large openings or gaps in a traffic stream. One approach focuses attention on a "major" stream of traffic in order to then study a variety of flow and storage situations which are created in a crossing or merging traffic stream. One of the more obvious merging problems is the one of a random arrival (in the minor stream) which waits for some combination of events associated with gaps in the major stream.

A second approach, and the one which has received the greatest attention to date, formulates the statistical process in the major stream in terms of certain observed conditions in the minor stream — say the wait of a random arrival at an intersection.

To the best of my knowledge, no one has as yet solved the problem which arises when the presence of a vehicle or some condition of vehicle delays in a minor stream directly affects the gap and block production process. The so-called impatience of a merging car driver, for example, may turn out to be the manifestation of a bluffing maneuver which makes a small headway into a large gap between vehicles in the major stream. There are many other theoretical questions which we have neglected in this paper but Weiss and Maradudin<sup>(14)</sup> have already made a careful survey of these problems.

The work of earlier authors has prompted me to make an attempt, in the remainder of this section, to relate the results and notation of this paper to those of earlier ones. Since the two articles by Mayne<sup>(7)</sup> and Weiss and Maradudin<sup>(14)</sup> contain complete bibliographies and since the authors make frequent reference to earlier results, I prefer to restrict my attention to their work. Table (2) identifies some of the equivalent notation in these papers.

TABLE 2

| Probability Density Distribution; Moment; Transform | This Article                           | Mayne                                      | Weiss and Maradudin <sup>o</sup>  |
|---|--|--|---|
| Inter-vehicle headway                               | $a(t); v; \sigma^2; \tilde{a}(s)$      | $x(u); m; -; -$                            | $\phi(t); \Delta; -; \phi^*(s)$   |
| Vehicle to beginning of gap headway                 | $b(t); v_b; \alpha_b^2; \tilde{b}(s)$  |  | $\eta(t); -; -; \eta^*(s)$  |
| Inter-gap headway                                   | $c(t); v_c; \sigma_c^2; \tilde{c}(s)$  |  | $\Gamma(t); \bar{t}_0; \bar{t}_0^2 - \bar{t}_0^2; \Gamma^*(s)$            |
| Block size  | $f(t); \psi; \sigma_f^2; \tilde{f}(s)$ |  |   |
| Gap size  | $g(t); v_g; \sigma_g^2; \tilde{g}(s)$  |  | $\rho(t); -; -; -$  |
| Size of unblocked period                            | $h(t); v_h; \sigma_h^2; \tilde{h}(s)$  |  |   |
| Random origin to first vehicle                      | $u(t); v_u; \sigma_u^2; \tilde{u}(s)$  | $\phi(t); -; -; -$                         | $\phi(t); -; -; \phi_0^*(s)$  |
| Random origin to beginning of gap                   | $v(t); v_v; \sigma_v^2; \tilde{v}(s)$  |  |   |
| Random origin to unblocked period                   | $w(t); v_w; \sigma_w^2; \tilde{w}(s)$  | $f(T); E(T); \text{Var}(T); L(s, T, f(T))$ | $\Omega(t); \bar{t}; \bar{t}^2 - \bar{t}^2; \Omega^*(s)$                  |
| Inter-vehicle and no gap                            | $a(t; x); -; -; \tilde{a}(s; x)$       | $-; h^{(1)}; h^{(2)} - h^{(1)^2}; h(s)$    | $\Psi(t); \Delta_1(T); \Delta_2(T) - \Delta_1^2(T); \Psi^*(s)$            |
| Random origin - no gap                              | $u(t; x); -; -; \tilde{u}(s; x)$       |  | $\Psi_0(t); \Delta_1^*(T); \Delta_2^*(T) - \Delta_2^{*2}(T); \Psi_0^*(s)$ |

<sup>o</sup> These relations are strictly correct only if Equation (20), Reference (14) holds, i.e., the minimum gap is T.

<sup>†</sup> The symbols  $\Delta$  and  $\Delta_0$  are used by these authors in several different contexts: as parameter of a probability density distribution, Equation (64), as random variable, Equation (70); as moment, Equation (71), (21), (15), as arrival time, Equation (82).

Mayne's<sup>(7)</sup> arguments are basically the following: If the conditional probability is  $w(t/T)dt$  that the wait of an arrival in the minor stream lies between  $t$  and  $t + dt$  given the headway from the randomly chosen arrival time to the passing of the first vehicle is  $T$ , then

$$w(t) = \int_0^{\infty} w(t/T) u(T) dT = U(x) \delta(t) \quad t = 0 \dots \dots \dots (7.1)$$

$$= \int_0^x w(t/T) u(T) dT \quad t > 0$$

is the marginal density distribution of wait for an unblocked period. The equivalence of the Laplace transforms of Equations (7.1) and (3.1) is easily shown by substituting  $A(x)$  for  $k$ ,  $v$  for  $m$ ,  $U(x)$  for  $F(0)$ ,  $x$  for  $I$  and  $\left[1 - \tilde{a}(s; x)\right]^{-1}$  for  $j(s)$  in Equation (5) of Mayne<sup>(7)</sup>. The transform

$$\tilde{w}(s) = U(x) + \frac{A(x)}{v s} \left[ \frac{1 - \tilde{a}(s; x) - A(x) e^{-sx}}{1 - \tilde{a}(s; x)} \right] \dots \dots \dots (7.2)$$

reduces to Equation (3.1) with the help of the identity of Equation (3.12).

Weiss and Maradudin<sup>(14)</sup> study the arrival time of a vehicle in the major stream (not necessarily the first one) given that a vehicle in the minor stream waits for this or a later arrival\*. A vehicle in the minor stream arrives at time zero. Let  $\omega(t)$

\* Again, it is important to point out that Weiss and Maradudin<sup>(14)</sup> concern themselves with the distribution of wait under more general gap acceptance criteria than the acceptance of an unblocked period.



be the density function of the conditional probability that a vehicle in the major stream arrives between  $t$  and  $t + dt$  given that the minor stream vehicle has not crossed. Either (i) this vehicle in the major stream is the first vehicle, or (ii) a vehicle arrives at time  $r < t$  and the headway between the next two vehicles is  $(t-r)$ . The renewal equation which expresses this statement

$$\omega(t) = u(t) + \int_0^t \omega(r) a(t-r) dr \quad 0 \leq t < x \quad \dots \dots (7.3a)$$

$$= \int_{t-x}^t \omega(r) a(t-r) dr \quad x \leq t \quad \dots \dots (7.3b)$$

can also be written as

$$\omega(t) = u(t; x) + \int_0^t \omega(r) a(t-r; x) dr \quad 0 \leq t \quad \dots \dots (7.4)$$

Since the wait of the minor stream vehicle is either zero or terminates with the beginning of a gap

$$w(t) = U(x) \delta(t) \quad t = 0 \quad \dots \dots (7.5a)$$

$$= \omega(t) \Lambda(x) \quad t > 0 \quad \dots \dots (7.5b)$$

one can obtain a solution for  $w(t)$  by substituting the solution of Equations (7.4) or (7.3) into (7.5).

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